Zeroes of Chromatic Polynomials

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Abstract

Let G be a graph and $P(G, \lambda)$ be the chromatic polynomial of G. We show that the maximal zero-free intervals for chromatic polynomials are precisely $(-\infty, 0), (0, 1), (1, 32/27]$, with chromatic roots being dense in $(32/27, \infty)$.

1 Introduction

Given a finite graph G without loops and multiple edges, $P(G, \lambda)$ denotes the chromatic polynomial of G. That is, given a nonnegative integer polynomial λ, $P(G, \lambda)$ gives the number of ways to color the vertices of G using λ colors such that no two adjacent vertices share the same color. While the chromatic polynomial loses its immediate meaning when λ is not an integer, it is nonetheless interesting to study the roots of $P(G, \lambda)$.

In order to study the roots of $P(G, \lambda)$, we first give some basic results of the chromatic polynomial (as found in [1]). From these, we show that $(-\infty, 0)$ and $(0, 1)$ are zero-free intervals of $P(G, \lambda)$. Next, we review Jackson's paper [2], mentioning some more sophisticated machinery, and give the result that $(1, 32/27]$ is also a zero-free interval of $P(G, \lambda)$. Finally, we show as Thomassen [3] did that chromatic roots are dense in $(32/27, \infty)$. Combining these results gives us that the maximal zero-free intervals for $P(G, \lambda)$ are $(-\infty, 0), (0, 1), (1, 32/27]$.

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2 Zeroes in $(-\infty, 1]$

It is obvious that any non-empty graph has no 0-colorings, and that any graph with at least one edge has no 1-colorings. Thus, 0 and 1 are zeroes of chromatic polynomials of graphs. To show that there are no other zeroes in the interval $(-\infty, 1]$, we must first give some intermediate results of chromatic polynomials. If G/e is a contraction of edge e, and $G \setminus e$ is a deletion of edge e, then we have

Theorem 1. $P(G, \lambda) = P(G \setminus e, \lambda) - P(G/e, \lambda)$.

Proof. Given a graph G, consider two vertices i, j with no edge between them. Any coloring of G falls into one of two categories: either i, j have the same color or they do not. Let e be an edge formed by connecting i, j. We can map all colorings where i, j have the same color bijectively to colorings of G/e. Analogously, we can map all colorings where i, j have different colors bijectively to colorings of $G + e$. Thus, we get that

$$P(G, \lambda) = P(G + e, \lambda) + P(G/e, \lambda)$$

Applying this identity in reverse (substituting $G \setminus e$ into G) gives us the desired result. □

Note that our proof only immediately gives us a relation between chromatic polynomials evaluated at a positive integer. However, since two degree n polynomials are identical if they agree on $n + 1$ points, the given expression holds for the polynomials themselves.

Since $G/e, G \setminus e$ both have fewer edges than G, we can use the equation obtained to make recursive or inductive statements about $P(G, \lambda)$. This kind of deletion-contraction argument is quite common in graph theory; in particular, we use it several of the following results:

Lemma 2. *The coefficients of the chromatic polynomial alternate in sign. That is, the coefficient a_m of λ^m is ≥ 0 if $n \equiv m \pmod 2$ and ≤ 0 otherwise.*

Proof. For an empty graph with n vertices (and in particular, no edges), $P(G, \lambda)$ is obviously λ^n since each vertex can be colored independently. Here, the result trivially holds.

Now, we induct on the number of edges of a graph. Given a graph G, assume that the result holds for any graph with $< |E(G)|$ edges. Selecting any edge e of G, we know the result holds for $P(G \setminus e, \lambda), P(G/e, \lambda)$. Applying theorem 1, it is clear that the result holds for $P(G, \lambda) = P(G \setminus e, \lambda) - P(G/e, \lambda)$. □

Theorem 3. *$P(G, \lambda)$ has no negative real roots.*

Proof. Suppose $P(G, \lambda) = \sum_{m=1}^{n} a_m \lambda^m$, where the coefficients alternate in sign by lemma 2. For $\lambda < 0$, then, we know that $(-1)^n a_m \lambda^m \geq 0$. Moreover, $a_n = 1$ (this is trivially for a graph with no edges; otherwise, it is a straightforward application of the deletion-contraction argument). Thus, for $q < 0$, $(-1)^n P(G, \lambda) > 0$, so $P(G, \lambda)$ has no negative real roots. □

Theorem 4. *$P(G, \lambda)$ has no real roots between 0 and 1.*

Proof. First, note that $P(G, \lambda)$ of a disconnected graph is the product of the chromatic polynomials of each connected component. Since multiplying polynomials does not give us roots that did not appear in the original polynomials, it suffices to consider connected graphs. We show that $(-1)^n P(G, \lambda) < 0$ for any $0 < q < 1$.

Consider any tree; there are λ ways to color an arbitrarily selected root. There are $\lambda - 1$ ways to color each child, since a child can be any color that is not the parent's. Thus, we see that for a tree, $P(G, \lambda) = \lambda(\lambda - 1)^{n-1}$. The statement that $(-1)^n P(G, \lambda) < 0$ can be easily checked for trees by binomially expanding $P(G, \lambda)$. Otherwise, the statement follows from a deletion-contraction argument, since $(-1)^n$ has a different sign from $(-1)^{n-1}$, and so for $P(G, \lambda) = P(G \setminus e, \lambda) - P(G/e, \lambda)$, $0 < \lambda < 1$, we have

$$(-1)^n P(G, \lambda) = (-1)^n P(G \setminus e, \lambda) + (-1)^{n-1} P(G/e, \lambda) < 0$$

We conclude that $P(G, \lambda)$ has no real roots between 0 and 1. □

Thus, combining theorems 3 and 4, we have the main result of the section:

Result 5. *$(-\infty, 0)$ and $(0, 1)$ are zero-free intervals for $P(G, \lambda)$, while 0 and 1 can be roots of $P(G, \lambda)$.*

3 Zeroes in $(1, \frac{32}{27}]$

In order to prove that $(1, 32/27]$ is a zero-free interval for $P(G, \lambda)$, Jackson defines a *double subdivision* as an operation on an edge ij: construct a new graph $G - ij$ by adding two new vertices and joining each new vertex to both i and j. Additionally, he defines a *generalized edge (triangle)* as either K_2 (K_3) or any graph that can be obtained from K_2 (K_3) by a sequence of double subdivisions. Given these tools, Jackson proves his main result that

Result 6. *$P(G,\lambda)$ is non-zero for all $\lambda \in (1, \frac{32}{27}]$.*

Proof. (Sketch; details omitted for the sake of length) The proof proceeds by contradiction: suppose there exist G, λ such that $P(G,\lambda)$ has a zero in $(1, 32/37]$. Jackson first shows in [2] that G is a generalized triangle, and based on several other claims, shows that $P(G,\lambda) \geq 0$. Next, breaking G down into several possible cases and justifying the upper bound 32/27, he generously manipulates expressions to prove that $P(G,\lambda) < 0$, contradicting the first result that $P(G,\lambda) \geq 0$. Thus, this gives the result that $P(G,\lambda)$ is non-zero for all $\lambda \in (1, 32/27]$. □

4 Zeroes in $(\frac{32}{27}, \infty)$

It remains to show that zeroes of chromatic polynomials are dense in $(32/27, \infty)$. To do this, we first state a major intermediate result of Thomassen:

Lemma 7. *Let λ_0, δ be real numbers, $\lambda_0 > 1, \delta > 0$. Assume there exists a graph G_0 having an edge xy such that*

$$|P(G_0, \lambda_0)| < (\lambda_0 - 1)|P(G_0/e, \lambda_0)|$$

Assume further that if $\lambda_0 > 2$, then

$$P(G_0, \lambda_0)P(G_0/e, \lambda_0) < 0$$

Then there exist real numbers λ_1, λ_2 and a graph H such that

$$\lambda_0 - \delta < \lambda_1 < \lambda_2 < \lambda_2 + \delta \qquad \text{and} \qquad P(H, \lambda_1)P(H, \lambda_2) < 0$$

In particular, $P(H,\lambda)$ has a root between λ_1 and λ_2.

Proof. (Details omitted for the sake of length) See [3]. □

Given this result, we are ready to prove Thomassen's main result:

Result 8. *If $\lambda_0 > \frac{32}{27}, \varepsilon > 0$, then there exists a graph G such that $P(G,\lambda)$ has a root in $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$.*

Proof. By lemma 7, is is enough to consider a graph G_0 with edge e such that

$$|P(G_0, \lambda_0)| < (\lambda_0 - 1)|P(G_0/e, \lambda_0)| \tag{1}$$

and (if $\lambda_0 > 2$),

$$P(G_0, \lambda_0)P(G_0/e, \lambda_0) < 0 \tag{2}$$

If $k \in \mathbb{N}$ such that $2 \leq k - 1 < \lambda_0 < k$, then we can let G_0 be K_{k+1}. Otherwise, if $\lambda_0 < 2$, then we do not need to consider (2). Some algebraic manipulation shows that the (1) is satisfied when G_0 is K_3 and $3/2 < \lambda_0 < 2$.

To narrow down the range λ_0 even more, consider the case where G_0 is $K_{2,3}$. It is known that

$$P(K_{2,3}, \lambda) = \lambda(\lambda - 1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7)$$
$$P(K_{2,3}/e, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$$

Given these chromatic polynomials, we can get that (1) is satisfied for $\lambda_0 \in [1.36, 3/2)$ when G_0 is $K_{2,3}$.

It remains to show the result for when $\lambda_0 \in (32/27, 1.36)$. Jackson in his proof of the results in [2] shows that there exists a minimal graph G_1 that is a generalized triangle such that $P(G_1, \lambda_0) > 0$. $P(K_3, \lambda_0) < 0$, so we can get G_1 from some G_0 by a double subdivision of an edge e in G_0, and by the minimality of G_1, we know $P(G_0, \lambda_0) < 0$. It follows from the definition of a double subdivision that

$$0 < P(G_1, \lambda_0) = P(G_0, \lambda_0)(\lambda_0 - 2)^2 + P(G_0/e, \lambda_0)(\lambda_0 - 1)^2$$

$P(G_0, \lambda_0) < 0$, so

$$P(G_0/e, \lambda_0)(\lambda_0 - 1)^2 > |P(G_0, \lambda_0)|(\lambda_0 - 2)^2$$

Since $\lambda_0 < (5 - \sqrt{5})/2 \approx 1.38$,

$$P(G_0/e, \lambda_0)(\lambda_0 - 1) > |P(G_0, \lambda_0)|$$

With this, we have covered all possible values for λ_0, and we are done.

□

5 The Main Result!

Result 9. *The maximal zero-free intervals for chromatic polynomials are precisely* $(-\infty, 0), (0, 1), (1, \frac{32}{27}]$, *with chromatic roots being dense in* $(\frac{32}{27}, \infty)$.

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7 Sources

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